Institut de Mathématiques et de Sciences Physiques
(IMSP)

The African School of Mathematics
Theme: Mathematical modeling for epidemology, ecology and conservation

Abomey-Calavi, 22-28 january 2018

INTRODUCTION TO MATHEMATICAL MODELING

Aboubacar MARCOS
abmarcos@imsp-uac.org

Contents

Introduction

This lecture is given at the The African Mathematical School organized from 22 – 28 january 2018, by the Laboratoire de Biomathématiques et d'Estimations Forestières (LABEF) in collaboration with the Institut de Mathématiques et de Sciences Physiques (IMSP) on the campus of university of Abomey-Calavi, (Benin). The lecture is an introductory course on modeling intended for graduate students and researchers in scientfic areas, at the first level in modeling training. It will be focused on some mathematical process of modeling and on a brief recall of dynamical systems as language of mathematical modeling. Therefore, participants at this course need to have some basic notions in ordinary differential equations such as solve first linear equations or nonlinear equation with seperate variables.

CHAPTER ONE

WHAT IS MATHEMATICAL MODELING

Models describe our beliefs about how the world functions. In mathematical modeling, we translate those beliefs into the language of mathematics. This has many advantages

1. Mathematics is a very precise language. This helps us to formulate ideas and identify underlying assumptions.

2. Mathematics is a concise language, with well-defined rules for manipulations.

3. All the results that mathematicians have proved over hundreds of years are at our disposal.

4. Computers can be used to perform numerical calculations.

There is a large element of compromise in mathematical modeling. The majority of interacting systems in the real world are far too complicated to model in their entirety. Hence the first level of compromise is to identify the most important parts of the system that seem relevant according to us with respect the model. This means that the character of a system is not only determined by the system itself but also by the observer of the system. Different observers can attach divergent meaning to a system. For example, an intensive livestock farm will, have a different meaning to a cattle farmer than to an environmental conservationist. This problem occurs frequently when group try to solve a problem jointly. If the mental or conceptual models are made explicitly then, it would be possible to avoid such a situation. A mental model is the image of the real systems that the observers forms in his head.
The second level of compromise concerns the amount of mathematical manipulation which is worthwhile. Although mathematics has the potential to prove general results, these results depend critically on the form of equations used. Small changes in the structure of equations may require enormous changes in the mathematical methods. So, a model is a selective simplification of the real world where only interactions relevant to the objectives of the model are considered. A mathematical model is a model where the interactions are governed by mathematical equations. The advantage of the mathematical model is that it allows the researcher to develop insight into the dynamic behavior of complex systems and to test these with the help of available data.

1.1 What objectives can modeling achieve?

Mathematical modeling can be used for a number of different reasons. How well any particular objective is achieved depends on both the state of knowledge about a system and how well the modeling is done. Examples of the range of objectives are:

1. Developing scientific understanding through quantitative expression of current knowledge of a system (as well as displaying what we know, this may also show up what we do not know);
2. Test the effect of changes in a system;
3. Aid decision making, including
   (i) tactical decisions by managers;
   (ii) strategic decisions by planners.
4. Prediction

1.2 Classifications of models

When studying models, it is helpful to identify broad categories of models. Classification of individual models into these categories tells us immediately some of the essential fact of their structure. One division between models is based on the type of outcome they predict: deterministic and stochastic
A deterministic model is a model where the state at a given time is totally dependent on the state at the previous time. The interactions are deterministic in the sense that always they predict the same outcome from a given starting point. The relationships ignore random variation, but are causal..
In a stochastic model, the relationships are not unequivocally determined but are probabilistic. The behaviour of the model cannot be predicted with certainly on the basis of the data available regarding the inputs . The model may be more statistical in nature and so may predict the distribution of possible outcomes.
A second division between models depends on how the variables of the models change: continuous or discrete.
The model is said to be discrete when the change of time in the model is discrete.
The model is said to be continuous when the change of time in the model is continuous.

1.3 Phases of modeling

It is helpful to divide up the process of modeling into four broad categories of activity, namely Building, Studying, Testing and Use. Although it might be nice to think that modeling projects progress smoothly from Building through to Use, this is hardly ever the case. In general, defects found at the Studying and Testing stages are corrected by returning to the Building stage. Note that if any changes are made to the model, then the Studying and Testing stages must be repeated.

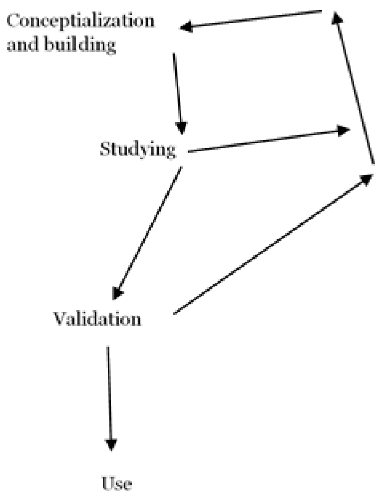

Figure 1.1:

This process of repeated iteration is typical of modeling projects, and is one of the most useful aspects of modeling in terms of improving our understanding about how the system works. We shall use this division of modeling activities to provide a structure for the rest of this course.
A pictorial representation of potential routes through the stages of modeling can be designed as follows

1.4 Conceptualization and Building

1.4.1 Getting started

The phase of the conceptualization of the modeling process is characterized by the choice of a problem area with a broad definition of a problem, the formulation of which is continuously more sharply defined during the process; in this phase the boundaries of the system must be defined as well. The basic elements are identified in terms of measurable variables. The types of variables can be categorized as follows

(i) Input-variables or exogenous variables: these variables come from the environment of the system. They can be separated into variables which are not changeable such as the weather conditions and into variables that are changeable (for example government measures) that we call policy variables or instrument variables.

(ii) Endogenous variables: There are variables come from the interior of the system. They can be classified between goal variables, constraint variables and intermediary variables. The goal variables are used to measure the objectives achievement and the constraint variables are used to determine at which extent the constraints have been satisfied. For example, earn a lot money (outcome is a goal variable) but without working more than 30h per week (working time is constraint variable).

(iii) Constants or parameters: variable that not change in value is a parameter. (for example, specific fertility of a specie, is a parameter)

1.4.2 System analysis

Having determined the system to be modeled, we need to construct the basic framework of the model. This reflects our beliefs about how the system operates. These beliefs can be stated in the form of underlying assumptions. Future analysis of the system treats these assumptions as being true, but the results of such an analysis are only as valid as the assumptions. Thus Newton assumed that mass is a universal constant, whereas Einstein considered mass as being variable. This is one of the fundamental differences between classical mechanics and relatively theory. Application of the results of classical mechanics to objects traveling close to the speed of light leads to inconsistencies between theory and observation. If the assumptions are sufficiently precise, they may lead directly to the mathematical equations governing the system. In population studies, a common assumption is that, in the absence of limiting factors, a population will grow at a rate which is proportional to its size. A deterministic model which describes such a population in continuous time is the differential equation.

$$\frac{dp}{dt} = ap$$

where $p(t)$ is population size at time t, and a is a constant. Solution of this equation by integration gives

$$p(t) = p(0)e^{at}$$

where $p(0)$ is population size at time zero. According to this solution, populations grow in size at an exponential rate. Clearly, not all populations grow exponentially fast. Since the differential equation arose from an interpretation of the assumption, we must look to the assumption for an explanation for this discrepancy. In this case, the explanation is the qualifier ”in the absence of limiting factors”. Most natural populations are subject to constraints such as food supply or habitat which restrict the range of sustainable population sizes. It is important that all assumptions are stated clearly and concisely. This allows us to return to them later to assess their appropriateness.
Another assumption which we made to obtain the differential equation was that growth takes place continuously. If the population consisted of discrete generations, we would have used the difference equation

$$d_{i+1} = bd_i$$

, where d_i is the size of the ith generation. This has solution

$$d_i = d_0 b^i$$

where d_0 is the initial population. Note that the solutions of the differential and difference equations can coincide at time $t = i$ if $d_0 = p(0)$ and $b = e^a$

Yet another assumption we have made is that the population behaves according to a deterministic law or a stochastic law. The model itself is defined by the population size $p$and the rates, for example birth rate, $b(p)$, and death rate, $d(p)$. In the deterministic case these define the rate of change of the population size by

$$\frac{dp(t)}{dt} = b(p) - d(p)$$

In the stochastic case these define (for suitably small δt) the probabilities of birth and death events, namely

	Event	Effect on population	Probability of Event
Birth	$p(t+\delta t) =$	$p(t)+1$	$b(p)\delta t$
Death	$p(t+\delta t) =$	$p(t)-1$	$d(p)\delta t$
No change	$p(t+\delta t) =$	$p(t)$	$1-b(p)\delta t-d(p)\delta t$

Having seen how many different assumptions have to be made to arrive at a simple model of population growth, we must proceed with extreme caution when formulating models of complicated systems. It is often wise to examine several different versions (e.g deterministic, stochastic) based on the same basic model as this improves confidence in any results derived from the model. .

1.4.3 Causal and flow diagrams

Where the system being modeled is more complex, we cannot simply jump from an assumption to an equation. We must be much more methodical, both when describing the system and when stating assumptions. Causal and flow diagrams are a visual aid to this end. Causal diagrams are graphical representattions of a model where all relevant variables and their mutual interactions are brought together throught the presence of positive or negative feedbacks in an overall picture. The flow diagram in their most basic form, consist of a series of boxes linked by a network of arrows. The boxes represent physical entities which are present in the system, whilst the arrows represent the way these entities inter-relate. Commonly used symbols in flow diagrams are: state or level variables source or sink channel of material flow channel of information flow control on rate of flow There are many ways of conducting a logical analysis of a system as a prelude of drawing a flow

Figure 1.2: Causal diagram

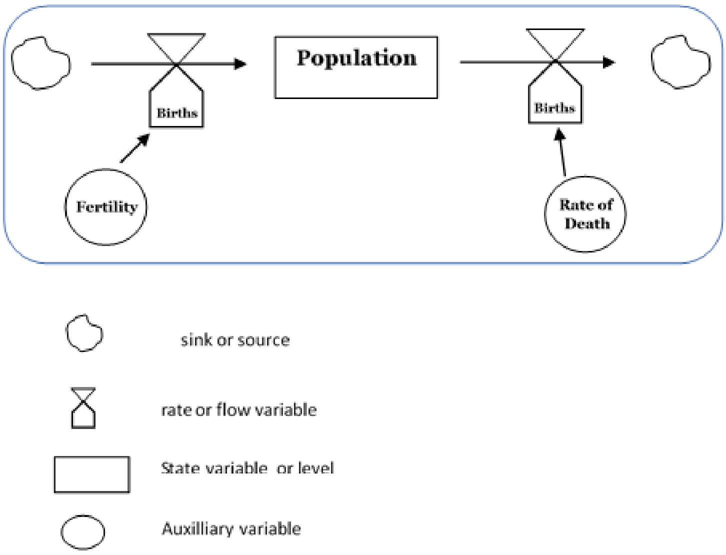

Figure 1.3: Flows diagram

1.4.4 Choosing mathematical equation

Here the system is described as a system of equation consisting of variables, parameters, using a specific computer language. If we are dealing with stochastic relationships, then besides variables and constants, the equations should also contains factor which indicates with which probability an event will occur. It is worth choosing these equations carefully - they may have unforeseen effects on the behaviour of the model.

1.4.5 Equations from the literature

It may be that somebody else has published an equation relating to the quantities you are interested in.This provides a good starting point, but it is necessary to proceed with caution. Problems encountered may include;
• equations derived from data with a range of explanatory variables which does not contain the range required for model application;
• experimental conditions (the environment) differ substantially from conditions to be encountered during model application;
• equations describe behaviour of the bulk of the data with no attempt to account for known departures at the end of range, or no account taken of variability.
Some areas of science are sufficiently well studied that appropriate forms of analysis have become standard. it is then relatively safe to assume that a similar analysis (and hence equation structure) carries over to similar problems. Often, equations in the literature will not be expressed in exactly the form required for the model. The dependent and explanatory variables in a regression may be transposed. Or an equation may describe the change in weight of an animal over time when the model requires knowledge about the rate of change. In either case, we may have to accept that the parameter estimates given are not necessarily the best ones for our purposes.

1.4.6 Deduction phase

In this phase calculation are performed based on the model. The result can be considered as the logical consequences of the structure of the model, the relationships between the variables and the input of the data. By means of careful analysis,the researcher can change or refine the model. Often, this is not a one-off process, but an iterative process in which the model cycle is followed several times. The calculations can be performed by solving the model equations analytically or numerically.

Analytically

There is much to be gained from obtaining an analytical solution to a model. This will allow us to perform all of the manipulations implied by the model with the minimum of diversion. Note that a full analytical solution for a stochastic model involves finding the distribution of outcomes, but we may feel satisfied if we can solve the equations for the mean and standard deviation. In general, obtaining an analytical solution is rarely a simple matter. In certain special cases, it is possible to obtain a mathematical solution to a system of equations. For example, in compartment analysis we often encounter linked differential equations of the form:

$$\begin{aligned} \frac{dx}{dt} &= a(t)x + b(t)y + c(t)z \\ \frac{dy}{dt} &= d(t)x + e(t)y + f(t)z \\ \frac{dz}{dt} &= g(t)x + h(t)y + k(t)z \end{aligned}$$

These equations are linear since there are not product terms of the form xx or xy on the right hand side. Not all of these equations can be easely handled by an analytical mehod because of the time dependency of the coefficients. When coefficients are constant, these differential equations can always be solved by a standard method. However, the method of solution is restricted to linear systems. If we were to consider a similar system, which contained just one non-linear term, then we could not apply this method. The analytical solution, if it exists, must be sought in other ways. If the model consists of just one differential equation, then there is a good chance that it has already been studied. Texts such as Abromowitz and Stegun (1968) contain a large number of standard integrals, and it is worth looking to see if the equation of interest is included. Similarly, analytic results can also often be obtained for linear stochastic models. However, when models contain nonlinearity, and most interesting models do, analytic results are typically harder to obtain than for the corresponding deterministic system. If the model is more complicated, and especially if the structure of the model is likely to be changed, then it is hardly worth even trying to find an analytic solution.

Numerically

When analytical methods are unproductive we can use numerical methods to obtain approximate solutions. Although they can never have the same generality as analytical solutions, they can be just as good in any particular instance. Numerical solution of model equations generally mimics the processes described in the model. For difference equations, numerical solution is exact since we can use the rules laid down in the equations to follow the evolution of the system. With a stochastic model, we can repeatedly simulate outcomes using a random number generator as described earlier, and combine a large number of simulations to approximate the distribution of outcomes. Differential equations provide a rather more difficult problem. The basic method is to divide continuous time into discrete intervals, and to estimate the state of the system at the start of each interval. Thus the approximate solution changes through a series of steps. For the simplest approximation method one can use is the the so called Euler's method. More sophisticated techniques are used in performing the Runge-Kutte types of integration. Fourth order Runge-Kutte is both commonly used and sufficiently accurate for most applications. It is always worth treating numerical solutions to differential equations with caution. Errors in calculating them may accumulate.

1.5 Studying the model

It is important to realize that the behaviour of a model can be described in two ways. Qualitative description provides an answer to questions about ”how”, whereas quantitative description answers questions about ”how much”. In general, qualitative behaviour will be the same for whole families of models and hence is amenable to general results. This contrasts markedly with quantitative behaviour, which is often only relevant to an individual circumstance. The qualitative behaviour of stochastic models is likely to show more diversity than the corresponding deterministic models. For example, different realizations of a stochastic population model may exhibit exponential growth and extinction. With stochastic models, therefore, it is important not only to describe the average behaviour but also to describe the range of types of behaviour.

1.5.1 Sensitivity analysis

The aim of sensitivity analysis is to vary model parameters and assess the associated changes in model outcomes. This method is particularly useful for identifying weak points and to estimate more accurately parameters of the model .These can then be strengthened by experimentation, or simply noted and caution taken in any application. Parameter estimation plays a less important role in modeling when one is more interested in the qualitative behaviour characteristics such as growth, stability, oscillating behaviour and the like than in predicting the exact value of a particular variable in a given year. Furthermore systems are often stable in the presence of negative feedback relationship and are not so sensitive to less accurate estimation of the parameters. Parameter estimation becomes important however if it appears that the behaviour of the model drastically changes as a result of small change in the value of the parameter. Apart from parameter value, the behaviour of a model can also be sensitive to the change of the initial values which are values assigned to the state variables at the beginning. Caution should be taken when parameter estimates are correlated, since if one parameter estimate is changed some of the others might have to be changed too.
The main question is now to know about how to construct the right type of equation and to determine the values of the parameters as accurately as possible. There is no exhaustive scheme. However, the following techniques are often used:

- Parameters can be estimated with or without linear regression analysis. The type of equation and parameters values one gives preference to, is dependent on whether one especially pursues a theoretical foundation of the model or a more empirical one. Regression analysis sometimes produces parameter values that are theoretically not plausible or even logically impossible.

- Using a sensitive analysis by which one can change variable and parameters in order to check:

  i) How sensitively the behaviour of the model reacts to small variations (errors) of the parameter values. There are the qualitative and the quantitative sensitivity analysis.

  ii) How do specific variables influence the goal variable; variables that have no influence could be eliminated.

  iii) Expert opinion if no data is available.

1.6 Validation of a model

Once we have studied our model and are satisfied with its performance, it is time to valid it by start testing the model against observations from the physical system which it represents. The validity indicates the extent to which a model provides a functionally adequate description of the system, this means a description that satisfies the goals of the model. Accordingly, it demands that the model includes:

(a) All variables that are relevant to the goal

(b) The appropriate policy variables: variables that could be manipulated by policy measures and subsequently influence the system.

(c) An appropriate time horizon (certain problems impact appears only after a reasonable time; if one is interested in the sustainability of agriculture in a given area and choose a time horizon of a few years, then there is risk that it is not possible to obtain insight in the developments that affects sustainability in the long run)

(d) The appropriate degree of detail, that is there is no point working in greater detail than the goal required

(e) Has the appropriate type of equations (linear, nonlinear, logarithmic, etc)

(f) Has assigned the convenient values to the parameters.

On can consider the validity in various ways

Theoretical validity
This is concerned with whether the model is theoretically sound: the equations must be theoretically correct and in agreement with tested knowledge.

Empirical validity
The model tested must give some outputs in agreement with the existing data. There is a strong reason for not using the same data as we used in parameter estimation - it will make us think that the model gives better predictions than it is really capable of. If the model should be used

to make predictions about the future, it is clear that such a performance is only possible if the described processes remain unaltered in the future.

Pragmatic validity of suitability
This is concerned with the suitability of a model for a particular problem. A well-known example is the model used for the Report of the Club of Rome. This model predicted a rapid depletion of mineral reserves. But it appeared that these predictions were inexact. Is it fair to say that the model has been of no value? No, because these pessimistic predictions of the model have induced many people to build a different opinion about the way in which we should deal with the world. In this respect, the model has brought a complete change in the consciousness of many and hence, enables us to avoid the disaster of the raw material depletion. If the model goal is the consciousness-raising, then despite the little empirical validity, it was nevertheless pragmatically valid

1.7 Using models

The method of presentation of a model to its eventual user depends to an extent on how much the user knows about the model. Since, in general, the user will know rather little about the details of the model, it is a good idea to present all relevant information in model output. This allows the user, not the programmer, to make the interpretation. It is almost invariably a good idea to check whether a prediction involves hidden extrapolation. Such extrapolation may be taking place either relative to the data used to build the model, or relative to the data used to test the model.

1.7.1 Predictions with estimates of precision

If the only output from a model is the prediction of some quantity, how can the user assess the accuracy of the prediction? Of course, this cannot be done, and the user is left in a take-it or leave-it situation. It would be better if the prediction were accompanied by an estimate of precision, such as a standard error or a confidence interval. These can be obtained from model studying or model testing. If we have investigated the effect of errors in parameter estimates when studying the model, we can estimate the precision of a prediction by summarizing the distribution of potential outcomes. This provides only a minimum estimate of error, since it takes no account of potentially erroneous forms of relationships used. Alternatively, the estimates of error might be carried through from prediction errors analyzed whilst testing the model. This is the best error to use, as it includes contributions from all possible sources. A direct application of estimates of prediction errors is the calculation of safety margins in feed relations. We know from experience that even when animals are fed an identical diet, they will grow at slightly different rates. Any model which describes the growth of the group will thus have to contain a stochastic element. If the target is for animals to gain weight at 1kg per day, choosing a ration from which the average gain is 1kg per day will mean that, on average, half the animals will not meet the target. In order for 95day plus 1.6 standard deviations (assuming a normal distribution).

1.7.2 Decision support

We now consider the task of embedding models in a economic framework through the cost effectiveness to assist decision-making. Costs are attached to various inputs to the model, such as animal feed or plant fertiliser. Input levels are chosen in a way that satisfies any constraints on the system. Conditional on the level of these inputs, the model is used to estimate biological outputs. The biological outputs themselves will have a financial value (e.g. sale price) attached to them. The difference between output and input values is then an estimate of gross profit. The aim of economic analysis may be to find the strategy which will be the most profitable. For simple models with a few control variables, this is usually not too difficult. When there are just two control variables, a graphical treatment is adequate. Numerical methods such as linear or quadratic programming may be applicable, but inaccuracies in the model predictions or economic conditions may render the extra number of significant figures calculated spurious. For complex systems with many control variables, optimisation may be a difficult problem. In special cases, it may be possible to adapt the problem so that linear or quadratic programming are applicable. Alternatively, a response surface could be fitted, treating profit as the variable to be predicted. Advanced methods such as dynamic programming may be needed to search through the web of interrelated decisions. Where the model outcome is stochastic, it is rarely possible to make statements like ”strategy A is always more profitable than strategy B”. Instead, we have to make statements about probabilities,like ”the average profit for strategy A is greater than the average profit for strategy B”. If this is true, would we always be wise to choose strategy A? The answer of course is no. It might be that strategy A has a 40 percent of chance of leading to bankruptcy, but a 60 percent chance of creating huge profits. Not everybody would be prepared to take the risks associated with strategy A.

CHAPTER TWO

OVERVIEW ON SOME MATHEMATICAL MODELS

Dynamical systems is the language of the mathematical models. So this part will give a brief survey of the differential equations of dynamical systems encountered in mathematical modeling

2.1 Ordinary Differential Equations (ODE)

Consider the variable $x(t) \in \mathbb{R}$, an ordinary differential equation (ODE) of first order read:

$$\frac{dx}{dt} = f(x;t)$$

$\frac{dx}{dt}$ can also write $x'(t)$ or $\dot{x}$ and describe the variation of x with respect the time t.

The autonomous/nonautonomous ODE .
The ODE is autonomous if $\dot{x}$ does not depend directly of t
$\dot{x} = f(x)$
The ODE is nonautonomous if
$\dot{x} = f(t,x)$

2.1.1 The autonomous linear/nonlinear ODE

Autonomous ODE

$$\dot{x} = f(x) = ax + b$$

with $(a;b) \in \mathbb{R}^2$

Nonlinear autonomous ODE (The Monod model.)

$$\dot{x} = \frac{a(b-x)x}{K-x}$$

2.1.2 Linear models analysis (the Malthus model or exponentiel model)

The malthus model is one of the first model of population dynamics (1800) . The hypotheses of this model are the following Consider a population of size $N(t)$ with a constante individual rate of change. How the size $N(t)$ change in the long run?
The variables $N(t)$ is the size of the population. The parameter r is the specific rate of change of the population.
The model equations are as follows

$$\begin{aligned} dN &= rNdt \\ \frac{dN}{dt} &= rN \end{aligned}$$

By solving the ODE, one has

$$N(t) = N_0 e^{rt}$$

The behaviour of the solutions $N(t)$ depends on the sign of r.

(i) increasing when $r > 0$

(i) constante when $r = 0$

(i) decreasing when $r < 0$

2.1.3 The Verhulst model or logistic model

The dynamics of population model of Malthus is not satisfactory because:
the rate of change r is independent of its size and suppose that natural ressources are illimitate. Such a model processes either to a demographic explosion of the specie or to its extinction .
Most natural populations are subject to constraints such as food supply or habitat which restrict the range of sustainable population sizes.
In the Malthus model, we have $r = b - d$ where b is the specific birth rate and d the specific death rate.
The model equations
The model hypotheses are:
The specific birth rate is constante and: $b = b_0$.
The specific death rate grows with respect te size of the population : $d = d_0 + \delta N$.
The specific rate of change of the population is then

$$r = b - d = b_0 - d_0 - \delta N.$$

The variation of the effectif dN during the infinitesimal time dt is :

$$dN = (b_0 - d_0 - \delta N)Ndt$$

$$\frac{dN}{dt} = rN(1 - \frac{N}{K}) \tag{2.1}$$

with $r = b_0 - d_0$ and $K = \frac{r}{\delta}$

Quantitative analysis of the logistic model

The solutions of (2.1) equations are as follows :

$$N(t) = \frac{N_0 K}{N0 + (K - N_0)e^{-rt}}$$

Four issues are possibles :

(i) If $N_0 = 0$, then $\forall t; N(t) = 0$.

(ii) If $K > N_0 > 0$, then $\forall t; \dot{N} > 0$ and $\lim\limits_{t \to +\infty} N(t) = K$.

(iii) If $N_0 = K$, then $\forall t; N(t) = N_0$.

(iv) If $N_0 > K$, then $\forall t; \dot{N} < 0$ and $\lim\limits_{t \to +\infty} N(t) = K$.

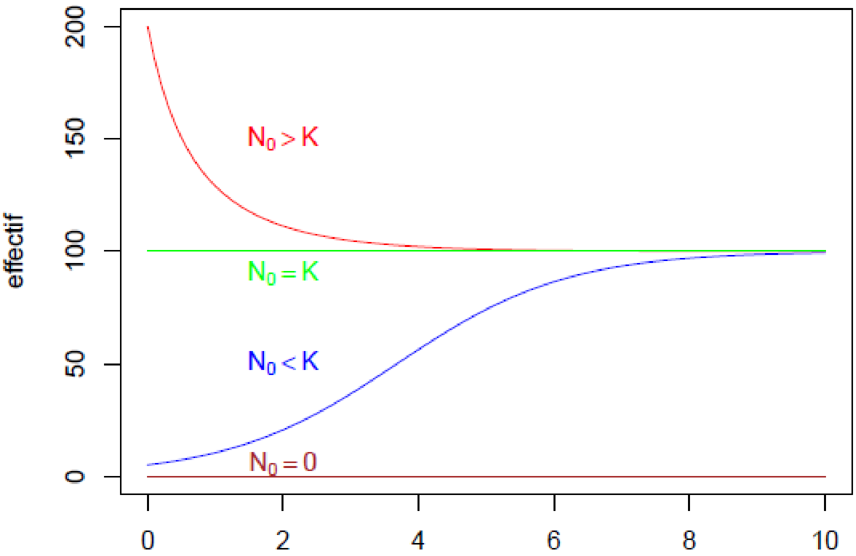

Figure 2.1: Graphic representation of the solutions

Proprieties of the system
Instead of giving the full quantitative study , we shall state the main properties such as.

1. The presence of invariants points
2. Behaviour of the models far from the invariant points
3. The shape of the solutions

Singular points or equilibrium points
The case of the logistic model
The equilibrium point $N*$ of the model is such that N is stationnary that is

$$\frac{dN}{dt}|_{N=N*} = 0$$

$$\frac{dN}{dt}|_{N=N*} = 0 \equiv rN * \left(1 - \frac{N*}{K}\right) = 0$$

There exit two equilibrium points $N_0* = 0$ and $N_1* = K$
Shall N tend towards the equilibrium points or move away ?

The behaviour of the system with respect the singular points
The logistic mode case
At equilibrium $N = 0$ and $N = 1$, $\frac{dN}{dt} = 0$
Out of the equilibrium points the sign of $\frac{dN}{dt}$ give the sens of evolution of N

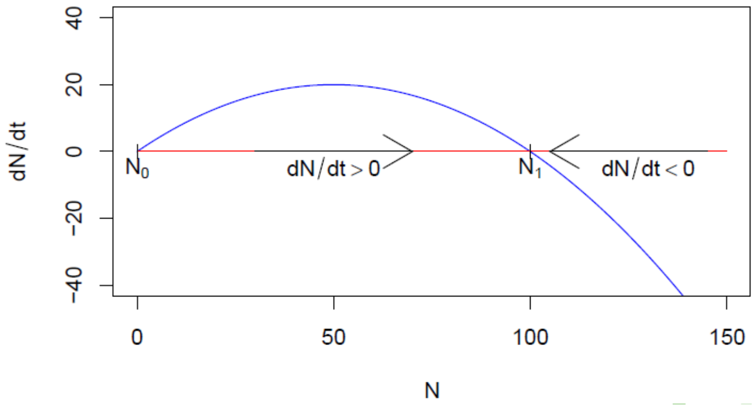

Figure 2.2: Behaviour of N

Phase portrait of the system
Logistic model case: the phase portrait of a dynamical sytems points out the singular points and the sens of variation of the variable in studying, around the equilibrium points .

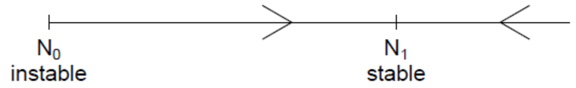

Figure 2.3: Phase portrait

2.2 Model of populations in Competition

General model
We consider two populations of effectifs x and y in competition governed by the system

$$\frac{dx(t)}{dt} = a(u)x(t) \tag{2.2}$$

$$\frac{dy(t)}{dt} = b(u)y(t) \tag{2.3}$$

where u denote the amount of restrictive resources for both populations. We can suppose that the ressources are strongly dependent on the populations $x(t)$and $y(t)$.

2.3 The predation: model of Lotka-Volterra

The model of Lotka-Volterra is the famous model of coexistence of two populations where one of both is a prey for the second (predator). The model is governed by the system

$$\begin{aligned}\frac{dx}{dt} &= px - qxy \\ \frac{dy}{dt} &= qxy - my\end{aligned}$$

Away from the predator, the prey X has a rate of grow constante and equal to P.
The preys are catched at rate proportional to the probability for a predator to meet a prey.
The rate of death of the predator is constant.
The equilibrium points can been performed by solving

$$\begin{aligned}px - qxy &= 0 \\ qxy - my &= 0\end{aligned}$$

By the analytical method, one gets that the solutions of the systems are all periodic solutions around the equilibrium points $x^* = \frac{m}{q}, y^* = \frac{p}{q}$.

2.4 Conclusion

It is beyond the scope of this introductory notes to cover the broad range of mathematical models. Mathematical modeling being a very broad subject such that in the literature each book cover only some aspects(Physics, economics, agriculture etc .) To learn more on the subject, the following books or lectures notes are recommended.

Bender, E.A. 1978. An introduction to mathematical modelling. Wiley, New York.
C. Laubry, Ecole du CIMPA Ouagadougou, juillet 1998, Modèle Dynamique Déterministe en Ecologie
Cross, M. and Moscardini, A.O. 1985. Learning the art of mathematical modelling. Ellis Horwood Ltd. Chichester.
France, J. and Thornley, J.H.M. 1984. Mathematical models in agriculture. Betterworths, London.
Turchin, P. 2003. Complex Population Dynamics, Princeton University Press, 3 Market Place, Woodstock, Oxfordshire, OX20 1SY.

Bibliography

[1] Abramowitz M. and Stegun I.A. , *Handbook of mathematical functions*, Dovers, New-York,1968

[2] Daniel L. and Glenn M.*An introductionnto mathematical modelling* , Bioinformatics and Statistics Scotland Given ,2008

[3] Richardson G. and Pugh III , *Introduction to system dynamics modeling*, Pegasus Communications, USA, 1999.

[4] Bontkes T. S. *Manuel pour le cours d'introduction à la simulation*, COSTBOX, IFDC-Africa, Lomé, Togo, 2001